A Note on Why There are No Even Harmonics in a Bipolar Square Wave

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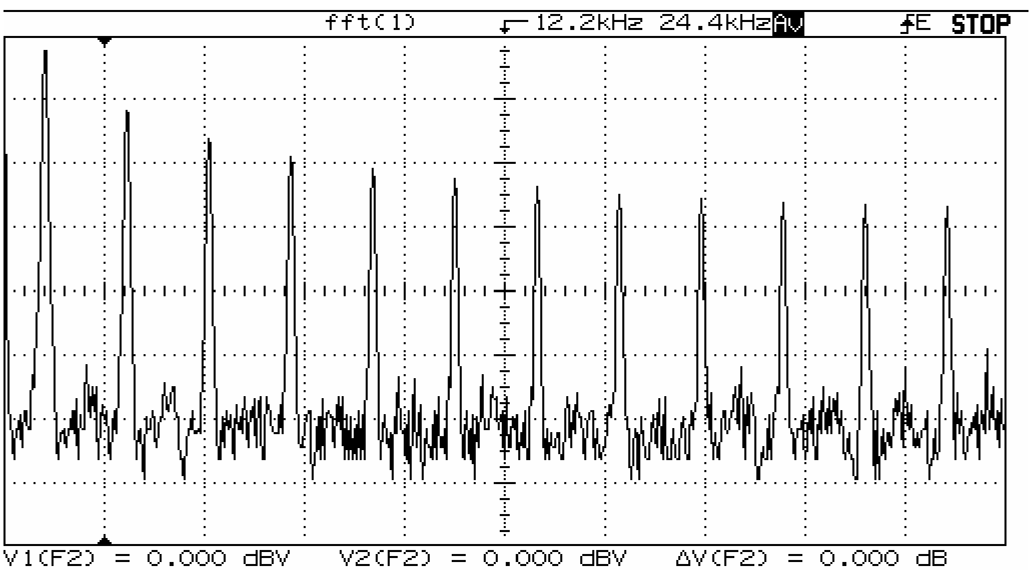
Exercise 2 of Laboratory 10 (Fourier Analysis) has us observe the FFT of a bipolar square wave. The exercise asks what property of the waveform makes these harmonics (i.e. the even harmonics) zero. Illinois is the Land of Lincoln, and Lincoln did note that you can fool all of the people some of the time. Such is the effect of this question. The question exposes a wide-spread misunderstanding of the Fourier expansion of even and odd (also known as symmetric and anti-symmetric) functions. I discuss this misunderstanding in this note.

The figure below shows the HP54600B FFT of a 1.00 kHz bipolar square wave. The frequency span is 0 to 24.4 kHz. The center frequency is 12.2 kHz. Only odd harmonics, i.e. 3, 5, 7,..kHz, of the fundamental, 1 kHz,. are present. The FFT of a triangle pulse would show the same feature.



Our task is to understand why only odd harmonics are present.

The Fourier series is applicable to periodic functions, i.e. functions which satisfy the condition

$$f(t) = f(t+T).$$

Define the set of frequencies $f_n = \frac{n}{T} \quad n = 1,2,3...$, and angular frequencies,

$$\omega_n = 2\pi\frac{n}{T} \quad n = 1,2,3...$$

Let $\omega = 2\pi\frac{1}{T}$. Following the notation of Experiment #10, the Fourier expansion of a general period function is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi \frac{n}{T} t) + \sum_{n=1}^{\infty} b_n \sin(2\pi \frac{n}{T} t) ,$$

or using the angular frequencies,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) .$$

The coefficients of these expansions can be obtained from the integrals

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{-T/2}^{+T/2} f(t)dt & a_0 &= \frac{1}{T}\int_{-T/2}^{+T/2} f(t)dt \\ a_n &= \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\cos(2\pi \frac{n}{T} t)dt , \quad \text{or} & a_n &= \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\cos(n\omega t)dt . \\ b_n &= \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\sin(2\pi \frac{n}{T} t)dt & b_n &= \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\sin(n\omega t)dt \end{aligned}$$

A change of variables gives an alternate form for the integrals.

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\omega t)d(\omega t) \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\omega t)\cos(n\omega t)d(\omega t) \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{+\pi} f(\omega t)\sin(n\omega t)d(\omega t) \end{aligned}$$

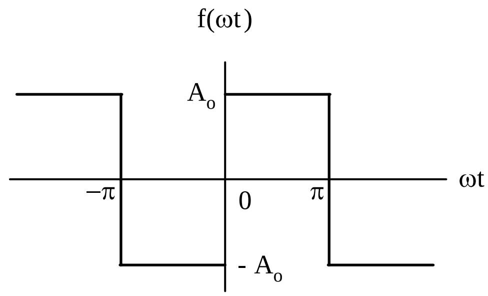
Note that the interval over which the integrals must be done is flexible. The same coefficients are obtained if the interval is displaced by a constant, c, as shown in the following equations.

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{c}^{c+T} f(t)dt \\ a_n &= \frac{2}{T}\int_{c}^{c+T} f(t)\cos(2\pi \frac{n}{T} t)dt . \\ b_n &= \frac{2}{T}\int_{c}^{c+T} f(t)\sin(2\pi \frac{n}{T} t)dt \end{aligned}$$

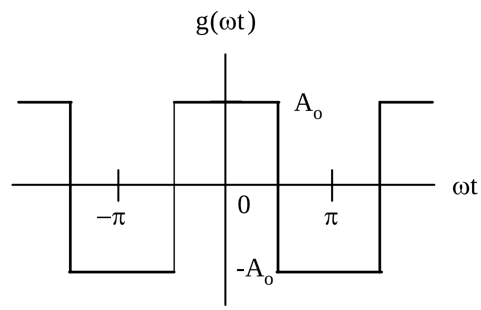
We need to do the integrals to find the Fourier series expansion coefficients, the a_n and b_n.

What is the misunderstanding that Exercise 2 exposes? The FFT display shows that only the $n = 1, 3, 5...$ Fourier components are present. It is incorrect to argue that only odd numbered Fourier components are present because the wave form is an odd function. Although a sine wave is an odd function, and a cosine wave is an even function, a bipolar square wave and a triangle wave could be either (or neither). More importantly, whether we have an even or odd function has nothing to do with whether its Fourier components are the $n = 1,3,5...$,or the $n = 2,4,6...$!

Let us first decide if a bipolar square wave is an even or an odd function. Two periods of a bipolar square wave are shown in the figure below.

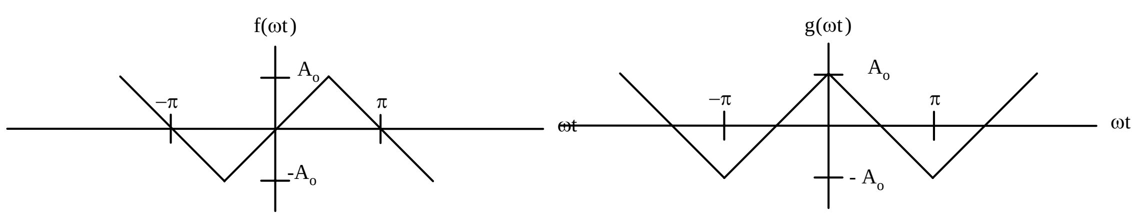


We note that the function has the property that $f(\omega t) = -f(-\omega t)$; this is an odd function. Consider, however, the bipolar square wave in the figure below, where in this case we also show two periods.

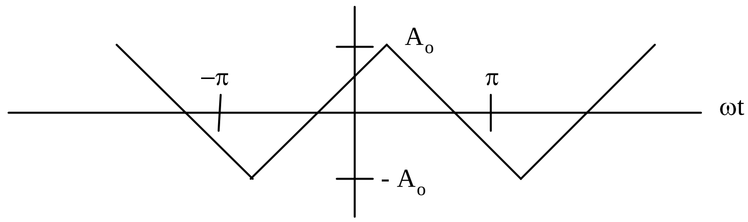


This function has the property that $g(\omega t) = g(-\omega t)$; this is an even function. The two functions are simply related: $f(\omega t) = g(\omega t - \pi/2)$. We could do a Fourier expansion for $f(\omega t)$ and $g(\omega t)$; the expansion for $f(\omega t)$ would have only sine functions $(a_n = 0, b_n \neq 0)$, and the expansion for $g(\omega t)$ would have only cosine functions $(a_n \neq 0, b_n = 0)$. For both expansions $a_0 = 0$; there is no DC component. Note that we make no statement as to whether $n = 1,3,5...$, $n = 2,4,6...$ or $n = 1,2,3...$ In principle, all Fourier components, $n = 1,2,3...$, could be present.

We could do the same exercise for a triangle wave, and we should realize that, depending on the phase, the function could also be even or it could be odd. The figure below should convince us.



If we choose the phase differently, we could have neither an even nor an odd function. This situation is shown below for a triangle wave. (We could make the same picture for the bipolar square wave, but the picture for the triangle wave is more convincing.)



The Fourier expansion of this triangle wave would have both sine and cosine terms, and, again, no DC component $(a_n \neq 0, b_n \neq 0, a_0 = 0)$.

The FFT shows the sum of the squares of the amplitudes for the cosine and sine terms, i.e. $a_n^2 + b_n^2$ on a log scale. All three versions of the triangle waves above would display the same FFT! Both versions of the bipolar square wave above would display the same FFT!

The Fourier expansions for all of these triangle and bipolar square waves have only $n = 1, 3, 5...$ These wave forms have another symmetry, namely, $f(\omega t + \pi) = -f(\omega t)$, or, $f(t + T/2) = -f(t)$. Note that this property is independent of the choice of phase. It is this property that produces only the odd harmonics. Trigonometry manipulations give

$$\cos(2\pi \frac{n}{T}(t + \frac{T}{2})) = \cos(2\pi \frac{n}{T} t + n\pi) = (-1)^n \cos(2\pi \frac{n}{T} t)$$
$$\sin(2\pi \frac{n}{T}(t + \frac{T}{2})) = \sin(2\pi \frac{n}{T} t + n\pi) = (-1)^n \sin(2\pi \frac{n}{T} t)$$

To satisfy the symmetry property $f(t + T/2) = -f(t)$, n must be odd.

In summary, many people confound even and odd functions with even and odd harmonics. Perhaps we should remember that even and odd functions are also called symmetric and anti-symmetric functions. If we only used the terms symmetric and anti-symmetric functions, we might not make the mistake. (Note that we never call some integers symmetric and others anti-symmetric.) Perhaps this distinction will help differentiate the two concepts.

For completeness we show below the connection between the symmetry of the wave form and the Fourier components.

A symmetric (even) function has only cosine terms. For a symmetric (even) function $f(t) = f(-t)$. Use the usual Fourier expansion.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi \frac{n}{T} t) + \sum_{n=1}^{\infty} b_n \sin(2\pi \frac{n}{T} t)$$

Let $t \to -t$ in the above expansion. Then

$$\begin{aligned} f(-t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(-2\pi \frac{n}{T} t) + \sum_{n=1}^{\infty} b_n \sin(-2\pi \frac{n}{T} t) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi \frac{n}{T} t) - \sum_{n=1}^{\infty} b_n \sin(2\pi \frac{n}{T} t) \end{aligned}$$

Use $f(t) = f(-t)$ to conclude $b_n = -b_n$, or $b_n = 0$. A symmetric (even) function has only cosine terms, and, possibly, a DC component.

An anti-symmetric (odd) function has only sine terms. For an anti-symmetric (odd) function $-f(-t) = f(t)$. Use the usual Fourier expansion.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi \frac{n}{T} t) + \sum_{n=1}^{\infty} b_n \sin(2\pi \frac{n}{T} t)$$

Let $t \to -t$ in the above expansion and change the sign. Then

$$\begin{aligned} -f(-t) &= -\frac{a_0}{2} - \sum_{n=1}^{\infty} a_n \cos(-2\pi \frac{n}{T} t) - \sum_{n=1}^{\infty} b_n \sin(-2\pi \frac{n}{T} t) \\ &= -\frac{a_0}{2} - \sum_{n=1}^{\infty} a_n \cos(2\pi \frac{n}{T} t) + \sum_{n=1}^{\infty} b_n \sin(2\pi \frac{n}{T} t) \end{aligned}$$

Use $-f(-t) = f(t)$ to conclude $a_0 = 0$, $a_n = -a_n$, or $a_n = 0$. An anti-symmetric (odd) function has only sine terms and, no DC component.